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Synthesis of nonuniformly spaced antenna arrays

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SYNTHESIS OF NONUNIFORMLY SPACED ANTENNA ARRAYS

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INTRODUCTION AND REVIEW OF PREVIOUS WORK

From an economical, as well as a practical, standpoint it is sometimes better to have an antenna composed of many small elements rather than one large antenna. The effective aperture of an antenna increases with the diameter, and the cost for doubling the diameter is far more than twice the cost of the smaller antenna [1]. Serious problems arise in controlling the surface geometry of a large antenna because of; 1) changing gravitational forces as the antenna is moved and 2) wind forces on the antenna.

The antenna array can have as good (or better) resolution and effective aperture area as a large antenna but does not have as many problems with gravitational and wind forces. The direction of the main beam of the array can be "steered" by appropriately controlling the phases of the currents in the cables that feed the array elements. However, the cost of the phased array may be large because electrically controlled phase shifters are expensive and the array may require one or more phase shifters per antenna element.

If the main beam is not required to be shifted in direction, the cost of the array may be much less than the cost of one single antenna that has the same radiation pattern as the array.

Until Unz's [2] paper in 1960 all published work on antenna arrays dealt with equally spaced elements. The beamwidth and sidelobe level of the array were altered by varying the amplitudes and phases of the currents fed to the antenna elements.

One of the common procedures for determining the amplitude factors of a uniformly spaced array is the Dolph-Chebyshev method [3, pp. 93-109]. The amplitudes are selected from the coefficients of a Chebyshev polynomial. Since some of the signal is dissipated in the attenuators that control the amplitude factors, additional antenna elements may need to be added to the original array to meet the gain requirements for the antenna. Instead of attenuators, amplifiers with various gains can be used to control the amplitudes, but increased problems with cost and stability are introduced.

The first published report (1961) that gave results of calculated patterns of nonuniformly spaced arrays was by King, Packard and Thomas [4]. By calculating antenna patterns for various preassigned element spacings, they demonstrated that the sidelobes can be reduced by spacing the elements unequally. This is a more efficient method than adjusting the amplitudes. Their sample array showed that grating lobes could be eliminated and that the beamwidth was nearly the same as for a uniformly spaced array of the same length. They found that monotonically increasing spacing usually gave the best patterns but did not show that it was a basic requirement.

In his first attempt to synthesize the spacings of a nonuniformly spaced array, Unz [2] used the Jacobi expansion to express the exponential phase factor as a sum of Bessel functions. The array factor $F(\theta)$ was then

$$F(\theta) = \sum_{n=-\infty}^{\infty} e^{jn\theta} \sum_{i=0}^{L} A_i \, J_n(kx_i)$$

where θ is the angle measured from broadside, x_i is the position of the ith radiator, A_i is an amplitude factor, and $J_n(kx_i)$ is an nth order

Bessel function. He set

$$f_n = \sum_{i=0}^{L} A_i \, J_n(kx_i)$$

and F (θ) became

$$F(\theta) = \sum_{n=-\infty}^{\infty} f_n \, e^{jn\theta} \quad .$$

If F(θ) were given, it could be expanded in a complex Fourier series with coefficients f_n. Then these coefficients are equated to the series of Bessel functions. No results of the procedure were given and Lo and Lee [5] later indicated that it is very difficult to get useful numerical results from this technique.

To study arrays with spacings larger than one wavelength, Unz [6] extended his previous theory by expanding the Bessel functions in an asymptotic series. Only the first term of the expansion was necessary if $|kx_i| >> \frac{1}{2}n^2$. From his analytical results he concluded that any arbitrary radiation pattern could not be approximated by an array with elements spaced at an average distance of four or more wavelengths.

In another short extension of his first paper, Unz [7] developed a magnetic field expression in the Fresnel zone for a nonuniformly spaced dipole array. The Fresnel zone pattern was expanded in terms of Gegenbauer polynomials and each coefficient in the expansion was a series of Bessel functions evaluated at the element positions. It was pointed out that the far field pattern can be found from the Fresnel region pattern by letting the distance from the array to the field point become infinitely large.

Fourier transform theory was applied to the synthesis problem by Butler and Unz [8]. The aperture was assumed to be infinite so that the limits on the integrals would be infinite. It would seem that a large error might be incurred, but no error analysis was performed and no example was shown to verify their assumptions. Nonuniform element amplitudes were required for an array to meet the design criteria and this could limit the theory's usefulness. No method was presented for finding the element spacing but if some spacing function were assumed, the amplitudes of the elements could be found.

In another effort by Butler and Unz [9] the array factor was written in terms of power in the region of the main beam. This power expression, in terms of matrix theory, was maximized and formulas were derived for finding the current distribution for a specified array function, but no technique was shown for finding the element spacings. An analytic expression was derived for the amplitudes if the elements were uniformly spaced, but no example was included that showed the quality of the design.

Unz [10] has also suggested creating an orthogonal set of functions from the terms in the array factor. The desired pattern function would be expanded in terms of the orthogonal functions and a set of equations would be found for the amplitudes of the array terms. However, if the element spacings were to be found for given amplitudes, say unity, a solution would be very difficult to find. Each element position would be tied up in a large number of trigonometric arguments. No examples were included nor was there any evidence that any design was attempted.

Another orthogonalization method by Unz [11] put a constraint on the argument of the pattern function so that $\{\cos ux_i\}$ became a set of orthogonal functions. The function u is $\pi\sin\theta$, and x_i is the position of the ith radiator in half wavelengths. The condition that had to be satisfied was

$$(x_i\pi)\tan(x_i\pi) = (x_m\pi)\tan(x_m\pi).$$

If one position were specified all other positions were determined. However, no method was shown for choosing an initial position to take advantage of the benefits of the nonuniformity of the element spacing. The distance between the adjacent elements given by the above formula was less than one wavelength.

If a pattern function to be synthesized were given, the only variables to be determined are the current amplitudes in each element. Consequently, for an array with a small sidelobe specification, the amplitude factors for a number of the elements would be small and the array would have low efficiency. However, if the additional gain were not needed, and the resolution is of primary concern, the array efficiency is not critical and the main loss is that of the additional hardware required to produce the correct amplitude distribution. No examples were presented in this article.

Sandler [12] expressed the nonuniformly spaced array as an equivalent uniformly spaced array (EUA). This was accomplished by expanding each cosine term in the array factor of the nonuniformly spaced array in a Fourier series. The individual expansions were then added term by term. The coefficients of the EUA were chosen to produce the desired array pattern by methods already developed for uniformly spaced arrays. These known coefficients were related to the element positions and were determined by a solution of simultaneous transcendental equations. Several of the equations are shown below:

$$A_o = \frac{\sin\mu_1\pi}{\mu_1\pi} + \frac{\sin\mu_2\pi}{\mu_2\pi} + \frac{\sin\mu_3\pi}{\mu_3\pi} + \cdots$$

$$A_1 = \frac{2\mu_1\sin\mu_1\pi}{\pi(\mu_1^2-1^2)} + \frac{2\mu_2\sin\mu_2\pi}{\pi(\mu_2^2-1^2)} + \frac{2\mu_3\sin\mu_3\pi}{\pi(\mu_3^2-1^2)} + \cdots$$

$$A_2 = \frac{2\mu_1\sin\mu_1\pi}{\pi(\mu_1^2-2^2)} + \frac{2\mu_2\sin\mu_2\pi}{\pi(\mu_2^2-2^2)} + \frac{2\mu_3\sin\mu_3\pi}{\pi(\mu_3^2-2^2)} + \cdots$$

$$\vdots$$

where the A's are the amplitude coefficients of the EUA, $\mu_m = \frac{2d_m}{R}$, d_m is the element spacing in wavelengths, and R is a scale factor. It is difficult to solve for the μ's from this set of equations and no attempt was made to design an array by this procedure. The expressions were used for a general discussion of the behavior of array patterns.

The feed system for the University of Illinois radio telescope was a nonuniformly spaced array designed by Swenson and Lo [13]. The desired pattern was the same as for a uniformly spaced array with a cosine-squared illumination function. The spacing function

$$y(x) = \frac{a}{\pi} \sin \frac{\pi x}{a} + x$$

for the nonuniformly spaced array with uniform illumination was obtained by integrating the original cosine-squared illumination function with the boundary conditions $y(0) = 0$ and $y(a) = a$. The aperture size is a, and the element position is x in wavelengths. Spacings were found by choosing a set of equally spaced y's and solving for the x's. The element spacing for the latter half of the array varied from 2/3 wavelength to 19 wavelengths with an average spacing of 1.85 wavelengths in this portion. In the first half of the array y was conveniently set equal to x.

By nonuniformly spacing the elements, only 276 of them were required instead of the 400 which are necessary for the same pattern if the elements were uniformly spaced with equal amplitudes. The pattern contained 206 sidelobes in a 0 to 90° region. For most of this region the sidelobes were below 0.032 and only exceeded this amount at a large distance from the main beam. The half-power beamwidth was 19 minutes of arc.

Lo [14] and Bruce and Unz [15] have both reported on a mechanical quadrature procedure. In Lo's method the Legendre-Gaussian quadrature was used. The element spacing was given by x_i as follows:

$$\int_{-a/2}^{x_i} f(x)\, dx = b(z_i + 1)/2$$

where z_i is the ith root of the Legendre polynomial of degree 2N (the total number of elements), a is the total aperture dimension in wavelengths, and b is defined by $\int_0^{a/2} f(x)\, dx = b$. The magnitude of the aperture distribution f(x) is not determined by this method and one must assume a distribution function to obtain the element spacing. An 80 element example was presented with a = 132 wavelengths. The following aperture distribution was assumed:

$$\begin{aligned} f(x) &= \cos^2 \pi x/a \qquad && \text{for } |x| \leq a/2 \\ &= 0 && \text{otherwise.} \end{aligned}$$

The beamwidth was approximately 0.57° and the sidelobes were very low near the main beam but increased to about 0.3 near the outer region. The pattern allowed a 90° scan angle under these conditions.

If a uniformly spaced array pattern has already been calculated, Harrington's [16] perturbation method can be applied on a lobe-by-lobe basis to reduce the level of the sidelobes. The perturbation was introduced by letting the element spacing be ε away from uniform spacing. Analytically this was expressed as

$$d_n = (\frac{n}{2} + \varepsilon_n)\ d.$$

If $\varepsilon_n u$ were small compared to one, the normalized field pattern reduced to

$$E = E_u - \frac{u}{N} \sum_n \varepsilon_n \sin n\frac{u}{2} .$$

where $u = \frac{2\pi}{\lambda} d \sin\theta$ and E_u is the pattern of a uniformly spaced array with element separation d. The previous equation was rearranged to give

$$\Sigma\ \varepsilon_n \sin n\frac{u}{2} = \frac{N}{u} (E_u - E) .$$

The ε_n's are then given by the formula for Fourier coefficients. However, the approximation is restricted to small $\varepsilon_n u$. The method works effectively when applied to lobes near the main beam. The lobe nearest the main beam was reduced from 0.2 to 0.1 on a normalized basis for a 12-element array. Reduction of the lobes near the main beam caused the lobes in the outer region to increase. In the 12-element array example, the outer lobes increased to 0.48 whereas a uniformly spaced array of 12 elements has a maximum sidelobe level of 0.22 except when a grating lobe occurs. Andreasen [17] pointed out that Harrington's technique is only effective if the average element spacing is less than about one-half wavelength. Under these conditions a uniformly spaced array would have no grating lobes.

Andreason [17] chose an initial array with preassigned positions and used a digital computer to calculate the sidelobe level. The position of one element was changed by the computer until a position was found that gave the lowest sidelobe level. When this position was found, the computer repeated the process for the next element in the array, and so forth. After all elements, except the center element, had been repositioned, the entire procedure was repeated. This process continued until the sidelobe level was below a certain number. It was found that no more than four complete loops were needed to arrive at the best array.

Arrays were designed with 11, 21, and 51 elements with an average spacing between adjacent elements of 2 to 3 wavelengths. The best sidelobe levels were 0.558, 0.374 and 0.3 for the 11, 21, and 51 element arrays respectively. Andreason tabulated parameters for equal-spaced Dolph-Chebyshev arrays with the same pattern data as the nonuniformly spaced arrays and in every case the required number of elements was less for the latter arrays, almost by a factor of two. Of course, with fewer elements the gain could not be as high as for the uniformly spaced array.

Maffett [18] formulated the problem as a continuous source distribution and integrated over the source aperture to give the array function. From the current density, a cumulative current distribution function was formed that incorporated the element positions. The integral was changed to a summation by applying the trapezoidal rule and dividing the interval into N spaces. The result was interpreted as an array factor of a

(2N+1)-element symmetric array.

An example was presented for 51 elements in which the cumulative current distribution was chosen. The spacing between adjacent elements was on the order of one wavelength. The sidelobe nearest the origin was about 0.06 in magnitude. The lobes steadily increased in magnitude to about 0.35 at $\sin\theta = 1$. A variable u was defined as $\sin\theta - \sin\theta_o$, where θ_o is the angle to which the beam is steered. For the range $0 \leq u \leq 2$ the maximum sidelobe level was about 0.4 for the 51-element array and the beamwidth at the half power points was 0.01 in units of u.

Baklanov et al. [19] attempted to make all sidelobe levels equal by setting the derivative of the array factor equal to zero to find the critical points and then equating the array factor to a constant at these critical points. This gave a set of simultaneous transcendental equations to solve. Since this was so difficult they avoided a direct solution and developed a system of quasilinear first-order differential equations by adding a differential value to each variable.

Various 8- and 17-element arrays were designed by solving their differential equations with a complicated numerical technique. The resulting patterns were quite good and had very nearly equal sidelobes that remained low. However, some of the elements were spaced as closely as 0.3 wavelength. This close spacing would cause strong mutual coupling that would change the pattern noticeably.

An "optimum" solution was shown by Brown [20] for 4- and 5-element arrays. No rigorous proof that the solution was optimum was given, but the patterns compared favorably with Chebyshev arrays of comparable size.

The symmetrical 4-element array factor was written as

$$E = \cos k_1\phi + \cos k_2\phi$$

where $\phi = \sin\theta$ and $k_n = 2\pi x_n/\lambda$. The two unknowns k_1 and k_2 were determined by specifying the levels of the first two sidelobes. The ratio of k_1 to k_2 determined the value of the first sidelobe and the second sidelobe was chosen to equal the first at $\phi = 1$. Thus both sidelobes had the same value in the visible range.

A comparison of derived results with a Chebyshev pattern showed very similar sidelobe levels, but the beamwidth of the Chebyshev pattern was 16% smaller. The first zero crossing of the nonuniformly spaced array was at $\phi = \sin\theta = 0.6$ and with this wide beamwidth only 2 sidelobes occured in the visible region $(-\pi/2 \leq \theta \leq \pi/2)$.

Ma [21] commented on Brown's [20] procedure to say that there was a special case of perturbational methods available for generalizing his results. A brief reiteration of the analysis of Baklanov et al. [19] was included and Ma continued with a slight modification which increased the spacing between adjacent elements of Brown's array. The perturbation started from a known equispaced uniform array. An example of an 8-element array was included that nicely illustrated the improved pattern. However, the increased spacings were still in the neighborhood of 0.5 to 0.8 wavelength. The sidelobe level was -17.22 dB or 0.138.

The effect of mutual coupling in nonuniformly spaced arrays was investigated by Allen and Delaney [22]. Theoretical and experimental studies were conducted on a nonuniformly spaced 16-element array. The spacing between adjacent elements varied from 0.5 wavelength to 1.051

wavelengths. The spacing was closest at the center and increased monotonically outward. The elements were dipoles placed one-quarter wavelength above a ground plane. The pattern was calculated by neglecting mutual coupling and then was compared to the experimental pattern. The first few sidelobe levels were 8 to 10 dB above the predicted levels in the absence of mutual coupling. To verify that mutual coupling was the cause of the discrepancey, the pattern was calculated as well as possible with the effects of mutual coupling included. Allen and Delany concluded that the agreement between this pattern and the experimental pattern was not excellent but the difference was in the proper direction. They conjectured that the remaining error was due to dipoles that were not as thin as they had assumed theoretically.

Ishimaru [23] converted the general term of the nonuniformly-spaced-array factor into an equivalent continuous source distribution by using the Poisson sum formula and introducing a "source distribution function". The latter gave the position of the nth element in the array when the variable in the function was set equal to n. The source distribution function was assumed to be

$$y(x) = x + 2\,\frac{A_1}{\pi}\sin \pi x \qquad\qquad A_1 < \frac{1}{2}$$

which allowed the pattern function in his formulation to be expressed in terms of Anger functions. This spacing function has the same form as the one used earlier by Swenson and Lo [13]. Approximations were made to keep the mathematics manageable but they also caused the sidelobes to increase as the position from the main beam increased. In an example of a 20-element array, the sidelobes increased from 0.06 near the main beam to about 0.22 at 90° from the main beam. The average element spacing was about 0.7 wavelength. When the average spacing was increased to about 1.2 wavelengths, the sidelobe level degenerated to about 0.45.

To find the element positions, tables of Anger functions were used in reverse to find the order and argument of the function for a specified array function behavior in the sidelobe region. This determined a value for A_1 and the number of elements N. y was set equal to n/N and values of x were found by a computer.

Ishimaru and Chen [24] extended the theory and presented a brief study of the Anger function and, in particular, investigated the influence on array parameters by the relationship between the order and the argument. An approximate relationship was found for the maximum sidelobe level and a detailed analysis was made of the gain.

A geometrical representation of the array factor was derived by Yen and Chow [25]. The sum of exponential terms (the array factor) was added geometrically and an integral expression for the factor was then deduced. Two types of zones of the array pattern were considered. A region of very low sidelobes was called a destructive zone and a region of higher sidelobes was called a stationary phase zone. It was shown analytically and by an 80-element example that for an exponentially spaced array the sidelobes were almost constant in amplitude in the stationary phase zones. The sidelobe level in the stationary phase zones varied from about 0.25 in one zone to 0.3 in another zone for the 80-element example. The main beam was very narrow and the sidelobes remained in the predicted regions for a large angular excursion.

In a second paper Chow [26] considered the previous plateaux problem by Yen and Chow [25] in reverse. He showed that in order to have low flat grating lobes the element spacing should be in exponential form. His procedure was to change the array factor to an integral expression by using Poisson's sum formula in the same manner as Ishimaru [23]. To create the stationary phase condition, the phase factor in the integrand was set equal to a constant. The results agreed with the previous work of Yen and Chow [25]. Relationships were found for arrays with nonisotropic elements and an example of a synthesis problem was included where the amplitude factors of the elements were specified.

Results of Yen and Chow [25] and Chow [26] were extended in Chow and Yen [27] to planar arrays. The elements were located on a lattice derivable from a conformal mapping of a uniform lattice. The derived array had exponential element spacings very similar to those of the linear exponentially spaced array discussed in their previous papers.

A graphical solution was reported by Bricout [28]. The array factor was written in the form

$$E = 2 \sum_{i=1}^{N} \cos a_i \psi$$

where $\psi = (2\pi L/\lambda)\sin\theta$, L is the half length of the array and $a_i = x_i/L$. The function was differentiated and set equal to zero to give

$$\sum_{i=1}^{N} (a_i \psi) \sin (a_i \psi) = 0 \ .$$

A curve was drawn of the function $f(x) = x \sin x$ and a drawing of a scale was included that was to be transfered to a transparent slide. The slide scale contained values of f(x) and when used in conjunction with a proportion scale f(x) could be read for any x. A root was found by adding values of f(x) corresponding to the a_i on a trial and error basis to detect a sum of zero. Once the root was found the transparency did not have to be moved to find the amplitude of a sidelobe at that value of x. An additional scale was provided for this.

Consideration was also given to arrays with amplitudes varying as the cosine function and all examples were for this configuration.

A method for designing a planar nonuniformly spaced array was proposed by Neustadter [29]. A second moment sum (SMS) was calculated for a predesigned amplitude-tapered array and equated to an SMS for a nonuniformly spaced array. Each term of the SMS was the product of the amplitude of that element and the distance squared from that element to a principal axis. There was an SMS for the x axis and an SMS for the y axis. The second moment sums were equated for only one small segment of each array at a time.

An example was included for a 35 dB Taylor taper 397 element array. The designed space-tapered array consisted of 217 elements of which 91 elements had unity amplitude and 126 elements had an amplitude of 0.599. The amplitude-tapered array had a gain of 1.46 times the gain of the space-tapered array but the latter required 45% fewer elements.

The technique of dynamic programming was used for array design by Skolnik et al. [30]. The process was accomplished on an element-by-element basis. The element positions were quantized and the outer pair of elements were fixed at the desired aperture size. The first element location was chosen anywhere in the aperture region and the second element was chosen similarly. For every position of the second element there was a "best choice" position for the first element to keep the sidelobes low. When the best second element position was found by trial and error, and consequently the best first element position, a third element was introduced. The second and third element positions were varied in the quantized positions until a "best choice" was found for the third element and for the second element. Since calculations already had been made on the best first element position for every second element position, the first element position was known. This process was repeated for each element in the array. This trial-and-error procedure required fewer computations than trying all possible combinations since consecutive element positions were taken two at a time, and once these positions were determined all previous element positions were determined.

Examples were included for 9- and 25-element arrays. The sidelobe level for a 9-element, 19λ aperture was -4.7 dB (0.582). This design allowed a 90° beam scan. It was found that as the quantizing increments became smaller, the sidelobe level was reduced. However, for increments less than 0.5λ, the improvement was small.

Marinos [31] operated with the array factor in terms of power rather than a field quantity. The excitation current i was broken into real and imaginary parts and for a broadside array the power pattern was

$$P(w) = \left[\int_{-\ell}^{\ell} i_r(z) \cos zw \, dz\right]^2 + \left[\int_{-\ell}^{\ell} i_i(z) \cos zw \, dz\right]^2$$

where $z = 2x/\lambda$, x is the element position along the array axis, $w = \pi\cos\theta$,

θ is an angle referenced to the array axis and $\pm\ell$ are the aperture limits $\pm L$ expressed in half wavelengths.

Three requirements had to be satisfied for the power function to be synthesized by this method. One requirement was that the function had to be separable into two positive functions $G_1^2(w)$ and $G_2^2(w)$ such that $P(w) = G_1^2(w) + G_2^2(w)$. By using Fourier transform theory the currents could be written in terms of the G's. A $P(w)$ that met all requirements was selected by trial and error and integrated in the transform expressions to get the real and imaginary values of the currents. Since these currents were continuous functions, and one is usually interested in discrete arrays, a Gaussian integration formula was used to convert the continuous array expressed in integral form to a summation form.

Results were given for 4 array designs. The design results were generally very close to the initially specified values. For a specified beamwidth of 40° and a sidelobe level of 20 dB (0.1), the design results were 40° and 19.6 dB. The element positions varied from a separation of 0.4 wavelength to 0.7 wavelength for a 6-element array. The excitations were $1.0\underline{/0°}$, $0.737\underline{/2.5°}$, and $0.222\underline{/-6.3°}$.

Barclay and Marinos [32] formulated the array factor initially in an integral form with a current amplitude distribution and an exponential phase factor so that the expression resembled the Fourier transform. However, the field expression was used rather than the power expression used by Marinos [31]. The current was broken into real and imaginary parts that were even functions so the array integral reduced to the Fourier cosine transform. The inverse transform was taken which produced real and imaginary expressions for the currents in terms of the radiation pattern. Gaussian quadrature techniques were then applied to change the continuous integral expressions to summations of discrete values.

The element spacings were not synthesized but were chosen to correspond to the zeros of a Legendre polynomial of order equal to the number of antennas. A field expresion was hypothesized that had a number of general parameters for producing whatever pattern one desired. The general form of the expression was a cosine function times a decaying exponential factor which controlled the sidelobe levels.

The quantities determined from this procedure were the current amplitudes of the array elements. Two included examples were an 8- and a 13-element array. The element spacings were on the order of 0.5 wavelength to 0.7 wavelength. The current excitations were complex, with values varying from 0.01 to 0.22 for the 8-element array. The sidelobes stayed below 0.1 for a typical 180° visible region and the half-power beamwidth taken from a plot of the 8-element array was about 16°.

An energy minimization technique in the sidelobe region was demonstrated by Galejs [33]. The element spacing d_n was assumed to be a quartic function of n. The sidelobe energy was minimized by adjusting each coefficient of the quartic equation independently (no minimization attempt was made by adjusting all coefficients simultaneously). The pattern of a 24-element array was similar to those of other techniques where the sidelobes were low near the main beam but increased markedly farther away.

Tang [34] introduced a method that utilized the pattern function of a previously designed uniformly spaced array with tapered amplitudes. The pattern functions of the uniformly and nonuniformly spaced arrays were each expressed in terms of a pattern of a continuous source distribution with piecewise uniform excitations. These pattern expressions were equated, which put a constraint on the element positions of the non-uniformly spaced array. By minimizing the difference in patterns near the main beam a simple expression for finding the element positions was obtained. Results of examples were included for designs from two Chebyshev amplitude tapered uniformly spaced arrays. The sidelobes near the main beam were low but increased greatly near the outer regions of the pattern. For a 10-element array the average element spacing was 1.075 wavelengths, the sidelobe level near $\theta = 0^\circ$ was 0.047, near $\theta = 90^\circ$ was 0.426, and the beamwidth was 5.4°.

Tang [35] continued his array study by comparing his previously selected array with an amplitude tapered array to show that nonuniformly spaced arrays must be thinned at the ends to have better sidelobe levels than those of a uniformly spaced array, or thinned at the center to produce worse sidelobe levels. Taylor's [36] line source distribution was used as the reference for a continuous array.

In a third paper Tang [37] presented numerical results on the beam-width and the operating region of the arrays he discussed previously. For a 50-element array with a 15 dB (0.178) sidelobe level and a 70 wave-length aperture, the beamwidth was 0.56' and the operating region was 23.9°. The operating region was defined as the region in which the side-lobe level stayed below the design level.

To complete Tang's study on the methods he previously introduced on nonuniform array design, Tang and Chang [38] considered the problem of optimizing the array gain. The gain was varied by adjusting the amplitude of each element. Several graphs showed the effect on gain by changing the number of elements and the design sidelobe level. This information could be used to design practical nonuniformly spaced antenna arrays.

Tang [39] gave a general discussion of the array problem from an approximation viewpoint and concluded that the problem required an entirely new effort. To restrict the infinite number of possible antenna arrays, he suggested quantizing the spacing and then formulated an expression for the total number of possible arrays for his quantization method. The optimum solution was to be found by calculating the pattern of each possible array.

Lo and Lee [40] presented a method of finding the secondary maxima of an array function if it contained only two terms representing four elements. The spacings were assumed to be an integral number of wavelengths. The maxima were found by minimizing the distance between the adjacent maxima of the two cosine terms. The details of the method of minimization were not explained in this paper, but reference was made to Hauptman and Karle [41].

A second method suggested by Lo and Lee in the same report was to approximate the cosine functions by triangular functions since the linear portions of the function would reduce computing time. For the practical cases of interest they then stated that the "frequency" of the end elements was much higher than the other elements so the highest sidelobe must be very near to one of the maxima (or minima) of this highest frequency component. An additional simplification was made by computing the pattern function over the maxima and the minima of the highest frequency component only. The element positions were quantized in half-wavelength steps and array patterns were calculated using the above approximations for all possible element positions in a specified aperture length. The last step in the procedure was to compute the exact sums of the array function at these locations. The approximated results were apparently within 1 dB of the true sidelobe levels so they felt that the approximations were fully justified for practical applications.

Larson et al. [42] formulated the nonuniformly spaced array problem when there is a variation in element size along the array as in a slotted waveguide. Parameters were defined and relationships between the variables were shown graphically, but no design technique was attempted. Graphs that were included showed how the grating lobe intensity changed with beam steering angle, fractional aperture size, and the degree of nonuniformity. The conclusion was that the use of unequal sized elements nonuniformly spaced could provide considerably increased beam-steering capability and smaller grating lobes.

Ma [43] built a theory around Haar's theorem which states that under certain conditions a sum resembling the array factor can be uniquely determined to approximate a function, the given array pattern, in the best "Chebyshev sense". Conditions were given that described a "Chebyshev system."

The process involved minimizing the error between the given function and the array factor by an iterative technique. The first step was to pick an element spacing, then find values of the given function at n points (n equals the number of elements). The element amplitudes were selected according to the given function at the n points by solving a set of simultaneous equations. Using these values in the array factor, an error function was written as the difference between the given pattern function and the array factor. The derivative of the error function with respect to $u = \pi \sin\theta$ was set equal to zero to find $n + 1$ extrema points of the initial error function. These $n + 1$ extrema points were used in the array factor with new unknown amplitude coefficients for the next iteration. A new error function was obtained by taking the difference of the desired function and the new array factor. The iterative procedure was repeated until the error function was equal in magnitude to a certain accuracy. The iterative process was proved to be convergent as long as the conditions for a Chebyshev system had been satisfied. Ma states that a serious limitation of this technique lies in how to choose the number of antennas and the element spacings so that a Chebyshev system is formed.

Matrix theory was used in a very general manner by Cheng and Tseng [44] to find an expression for an optimum value of a quantity in the sense of a maximum or minimum. In the latter half of their paper they applied their theory to maximize the gain of arbitrary arrays and an explicit expression was found for the optimum excitations of the elements. The only results shown were for the amplitudes of two 8-element uniformly spaced endfire arrays.

A valuable paper in January 1966 was Lo and Lee's [5] study of space-tapered arrays. The difficulty of the problem was emphasized by noting that if the element positions were subject to optimization, the problem becomes highly nonlinear and that the location of the highest sidelobe does not vary continuously with element positions.

To compare the results of previous design techniques with what could be considered an "optimum" array in the sense of the least sidelobe level for a given number of elements and beamwidth, an exhaustive study was made of a few small arrays. The 9-element symmetric array received the most attention. A 19.5 wavelength aperture was considered with quantized element positions every quarter wavelength. With one element always at the center and a fixed element on either end, there were 7770 different ways of placing the remaining 6 elements with no two elements occupying the same position. Pattern functions were calculated for all of these cases in increments of 1/76 out of 1. The lowest sidelobe level obtained was 0.5145 and the average element spacing was 4.75 wavelengths. Twenty-three arrays out of the possible 7770 had sidelobe levels between 0.5 and 0.6. From the 23 best arrays no conclusions could be drawn as to a method of element spacing since the positions varied so drastically among the arrays. However, looking at the 23 cases as a whole, it was observed that there were more elements closer to the center of the array than at the ends.

The best array was compared with results of authors mentioned previously to see how close to a true optimum solution the designs were. Skolnik _et al_. [30] considered a 9-element array in their dynamic programming procedure. However, after calculating 648 cases the lowest sidelobe level obtained was 0.583. The sidelobe level of a 9-element array pattern calculated with Ishimaru and Chen's procedure [24] was worse than 58% of all the cases considered in Lo and Lee's study. Another comparison was made with a design based on Chow's [24] technique. For both a 10-element array and a 26-element array, Chow's sidelobe levels were higher than the lowest levels found by Lo and Lee.

Lo and Lee concluded that space-tapered arrays have a larger chance to produce low sidelobe levels than a non-tapered array, but many low level arrays are not space-tapered.

Shih [45] reported on a synthesis method using the lambda function. To arrive at a lambda function expression, the Chebyshev-Gaussian quadrature and Hankel transform were used. The element positions were not synthesized, but were spaced according to the zeros of a Chebyshev polynomial and the synthesis was applied to the amplitudes. To use the technique the desired pattern function must be expanded in terms of the lambda function. This expansion is difficult since the lambda functions do not form an orthogonal set over the desired interval. This problem was overcome by taking the Hankel transform of the desired pattern function and expanding the result in a power series. A relationship was then found for the coefficients of the lambda function expansion.

An example problem was solved for a 12-element array. The assumed pattern function was of the Taylor [36] type and the sidelobe level was chosen as 0.1. The synthesized pattern compared very favorably with the prescribed pattern. The amplitude ratios were quite severe and would be a decisive argument for not using such a technique. The amplitude ratio was smallest at the end (0.05 out of 1.0) and increased to 1.0 at the center.

In a technique presented by Strait and Cheng [46], small variations from the uniformly spaced array were represented by amplitude factors. If the element amplitudes were unity the array factor for an even number of elements N is

$$E = \frac{2}{N} \sum_{i=1}^{N/2} \cos\left[\frac{2i-1}{2} + \Delta_i\right]\psi$$

where

$$\psi = \frac{2\pi d}{\lambda} \sin\theta$$

d = nominal interelement spacing

Δ_i = spatial variation from uniform spacing

For small Δ_i the authors show that

$$E \simeq \frac{2}{N} \sum_{i=1}^{N/2} \left\{\cos(2i-1)\frac{\psi}{2} + \frac{2\Delta_i}{\pi}\left[\cos(2i+1)\frac{\psi}{2} - \cos(2i-3)\frac{\psi}{2}\right]\right\} .$$

The usefulness of this equation arises from the Δ_i appearing only as an amplitude variation and not as a spatial variation. The above expression was equated to a Dolph-Chebyshev array factor which was expressed as

$$T = \frac{2}{N} \sum_{i=1}^{N/2} A_i \cos(2i-1)\frac{\psi}{2} \quad .$$

The two array factors were matched as closely as possible by equating coefficients of similar cosine terms. A set of linear equations resulted that related the Dolph-Chebyshev amplitude coefficients to the variation in element spacing.

An example was calculated for a 16-element array with a design sidelobe level of 20 dB (0.1). The synthesized pattern had a sidelobe level very close to 20 dB and the half-power beamwidth was about 6°. The average element spacing was about 0.48 wavelengths.

Meyer [47] analyzed the problem by using the Fourier transform and associated theorems. The array factor was easily changed into integral form by placing the Dirac delta function $\delta(x-x_n)$ in the integrand. The amplitude factor and the spacing factor were each represented as the Fourier transform of another function. The complete array factor was then written as a convolution integral. After showing that the array factor could be represented by a number of other forms also, he demonstrated the application of one of his formulas by choosing the same position function as Ishimaru and Cheng:

$$x = y + \frac{2A_1}{\pi} \sin \pi y ,$$

but no conclusions were drawn from this result other than to point out the similarity with a result of Ishimaru and Cheng.

Additional manipulations were carried out with the end result being an expression for the array factor in terms of many multiple sums of Bessel functions. This method of pattern calculation appears far more complex than using the original summation form for the array factor.

For completeness, the statistical approaches should be mentioned. However, they usually require several hundred elements and arrays of that size were not of interest at this time so reviews were not made. Designs based on statistical methods will be found in references [48] through [54].

An experiment by Lo and Simcoe [55] verified extremely well statistical results predicted earlier by Lo [52]. The experimental technique used was the "holey-plate" method which has been described by Skolnik [56]. The antenna array was modeled with a conducting circular screen about 56 wavelengths in diameter perforated with small holes to simulate the antenna elements. Two sample planar arrays were constructed, each with 210 elements, and the diameter of each hole was about one quarter wavelength. The frequency used was 71.25 GHz. The experiment showed that mutual coupling could be ignored if the average spacing were not too small. The agreement between measurement and the theory in sidelobe level, sidelobe distribution and half-power beamwidth was extremely good.

In many of the papers presented previously, mathematical difficulties were encountered in the deterministic methods that made the calculation of element positions completely impractical, or approximations were made that would lead to less desirable array patterns. Some methods required an element-spacing function to be specified and only the element amplitudes were synthesized. Frequently no procedure was outlined for finding this element spacing function. In some instances no results were included, so evaluating the method was difficult.

Before proceeding with the design technique presented in this thesis, the nomenclature used in array theory will be discussed.

PROBLEM DEFINITION

The geometry of the linear antenna array to be considered in this thesis is shown in Fig. 1. The spacing between adjacent elements in unequal but the array is assumed to be symmetrical about the origin. The elements are considered as isotropic radiators since the pattern for an array of similar directive elements is simply found as the product of the pattern function of an array of isotropic radiators and the pattern function of an individual pattern element [3, pp. 66-76]. The observation point is assumed to be a large distance from the array so that the radiation is approximately in the form of plane waves whose directional rays are parallel along the path between the array and the field point of interest. The electric field has harmonic time dependence and can be expressed as $E_o \cos(\omega t - \beta r)$ where E_o is a constant, ω is the angular frequency, $\beta = 2\pi/\lambda$, and r is the distance along the propagation path.

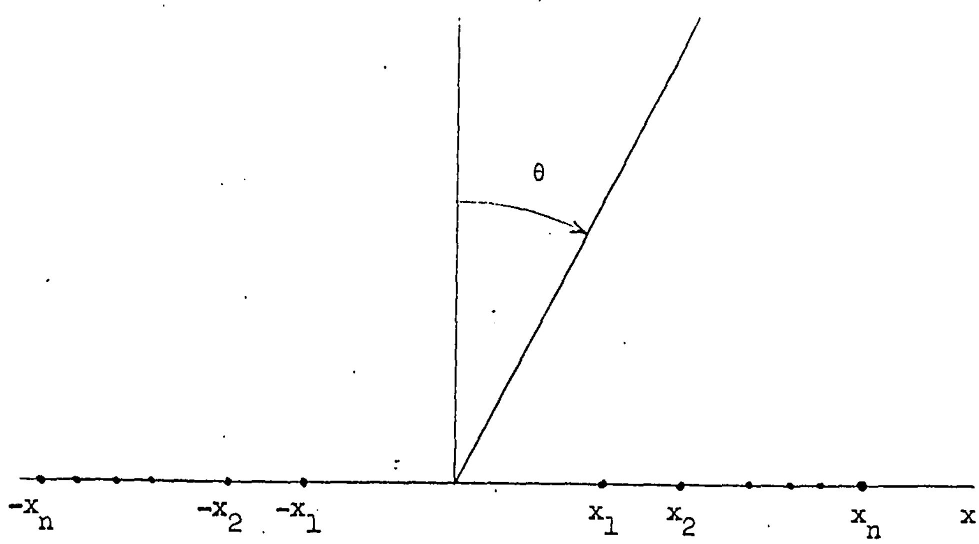

Fig. 1. Array geometry

The antenna pattern is caused by interference of the waves from individual antenna elements. The distance that determines the amount of interference is the difference between the propagation path length from an individual antenna element and the phase reference, which will be the origin. This distance for an element located at x_1 is $x_1 \sin \theta$, for an element at x_2, $x_2 \sin \theta$, etc. The phase difference between an element at the origin and an element at x_1 is $\beta x_1 \sin \theta$. For convenience the wave function is expressed in an exponential form and the time dependence is dropped. The phase relationship between the element at the origin and the element at x_1 would then be $e^{+j\beta x_1 \sin \theta}$.

For 2N+1 elements the field pattern can be expressed as

$$E(\theta) = A_o + \sum_{n=1}^{N} A_n (e^{-j\beta x_n \sin \theta} + e^{+j\beta x_n \sin \theta})$$

$$= A_o + 2 \sum_{n=1}^{N} A_n \cos(\beta x_n \sin \theta) \tag{1}$$

where A_n is the amplitude factor for the nth pair of elements. If the amplitude factors are identical, the expression reduces to

$$E(\theta) = A_o + 2 A_o \sum_{n=1}^{N} \cos(\beta x_n \sin \theta) . \tag{2}$$

Several simplifications can be made to permit easier manipulation. Let $d_n = x_n/\lambda$ and $v = 2\pi \sin \theta$. Then Equation 2 reduces to

$$E(v) = A_o + 2 A_o \sum_{n=1}^{N} \cos d_n v . \tag{3}$$

Dividing by the maximum value of E(v) reduces Equation 3 to a normalized expression F(v), which is called the array factor:

$$F(v) = \frac{E(v)}{A_o(2N+1)} = [1 + 2 \sum_{n=1}^{N} \cos d_n v]/(2N+1) \ . \qquad (4)$$

The synthesis problem of interest is to determine the d_n's when the array factor $F(v)$ is specified. A solution for the d_n's when the number of antenna elements and the sidelobe level are specified is presented in this thesis. The magnitude of the sidelobes is of primary importance, but the functional variation in the sidelobe region is not significant. The cosine displacement design method presented in this thesis will be explained for an array of 5 elements which is sufficient to convey the basic design principles. Following the theory are design examples for 5-, 7- and 9-element arrays. The last section contains suggestions for extending the theory to larger arrays than previously discussed and to arrays with nonisotropic elements.

The design problem is more difficult than one may realize from a brief introduction. Well known optimization techniques, such as minimizing the mean square error, are not applicable to this problem. The position of the highest sidelobe does not vary continuously with the element positions, and causes additional complications. A highly nonlinear problem is created when an attempt is made to determine the element positions by optimizing the array function. In addition, there is no particular "best" functional form to which the array function can be optimized for a practical solution. An impulse function provides the ideal directional properties, but would be completely impractical for array design since an extremely large number of elements would be required.

Another complication arises in introducing the design specifications into the problem since there is no closed function to which they can be applied. One has only the series of cosine functions to work with. Using this series, it is very difficult to obtain a workable relationship between the beamwidth or sidelobe level and the element positions.

THEORY OF COSINE DISPLACEMENT METHOD

Theory of 5-Element Arrays

Consider an array of five ($2N+1 = 5$) elements. From Equation 4 the array factor is

$$F_5(v) = \frac{1}{5}(1 + 2 \cos d_1 v + 2 \cos d_2 v) \ . \tag{5}$$

Let $d_1 = A$ and $d_2 = A+B$ with B greater than zero. Then

$$F_5(v) = \frac{1}{5}[1 + 2 \cos Av + 2 \cos(A+B)v] \ . \tag{6}$$

If high sidelobes are to be avoided, the peaks of the two cosine functions should never coincide at any point except $v=0$ within the range of v, which is $- 2\pi$ to $+ 2\pi$. Since the cosine is an even function, the range of v can be reduced to 0 to 2π. Fig. 2 shows a sketch of cos(Av) and cos(A+B)v with B not equal to A.

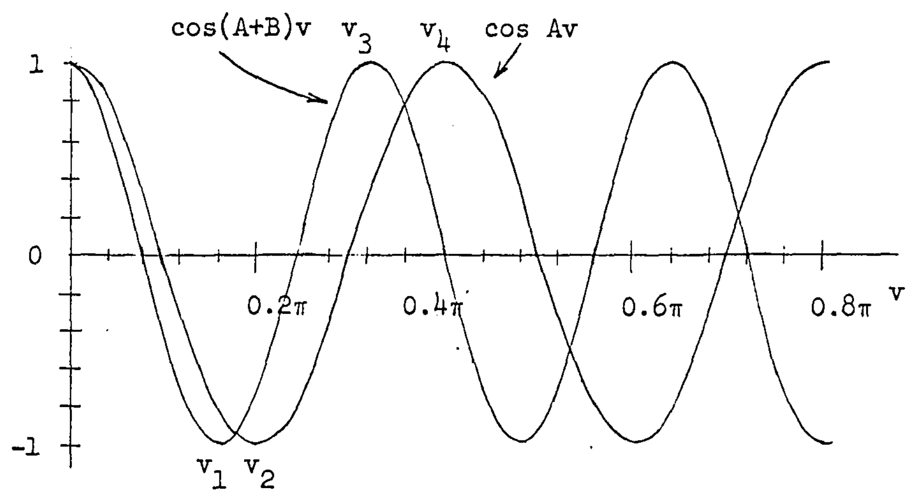

Fig. 2. Cosine functions for an arbitrary 5-element array

The displacement between the two curves can be expressed as Bv which would be equivalent to a phase shift if the two cosine functions had equal periods. By controlling the displacement between adjacent peaks of the individual cosine functions, the magnitude of the sidelobes can be regulated. The minimum points of the cosine functions will be referred to as negative peaks. One can observe in Fig. 2 that as v increases from zero, the displacement between the peaks increases. When $(A+B)v$ exceeds Av by π radians, the distance between two peaks starts to decrease, and as v continues to increase, the functions will coincide at some point where $Bv=2\pi$. Beyond this point the displacement cycle repeats.

The curves can be prevented from coinciding within the range being considered, if the point of coincidence occurs for v greater than 2π radians which means that B is less than one.

Because of the cyclic properties of the displacement, the distance $2\pi - Bv_1$ between two curves is the same as Bv_1. However, if the curve of $\cos(A+B)v$ is "leading" in the former case, it is "lagging" in the latter. The sum of $\cos(Av)$ and $\cos(A+B)v$ is relatively large near v_1 since the two functions disperse beyond this point, and will be the largest sum in the region $v_1 \leq v \leq 2\pi$ if

$$2\pi - Bv_1 = 2\pi B , \tag{7a}$$

where $2\pi B$ is the displacement at $v=2\pi$. This equation requires the curves at $v=2\pi$ to be as far apart as the same curves at $v=v_1$. In terms of degrees this says, for example, two curves an angular distance of $45°$ apart at v_1 are required to be $315°$ apart at $v=2\pi$.

Replacing v_1 by $\pi/(A+B)$ in Equation 7a and solving for A gives

$$A = \frac{B(2B-1)}{2(1-B)} \quad . \tag{7b}$$

This relationship must be satisfied if the two curves are not to coincide in the region $0 < v \leq 2\pi$. A consequence of this formula is that B is restricted to values between 0 and 1.

Minimum Sidelobe Design Problem

The pattern with the lowest sidelobes that has negative peaks for both cosine functions in the region $0 \leq v \leq 2\pi$ occurs for $A = \frac{1}{2}$, as can be seen by sketching the functions. For A somewhat less than 1/2, the first negative sidelobe moves outside $v=2\pi$. When the first negative sidelobe continues to move beyond $v=2\pi$, the main beam encompasses the entire visible region $(0 \leq v \leq 2\pi)$ which is not desirable for a directional antenna. On the other hand, when the element-spacing increases, the peaks move toward the origin. This decreases the beamwidth, but increases the sidelobe level since the peaks are closer together.

If A=1/2, Equation 7b gives B as $\pm 1/\sqrt{2}$. Only the positive value is used since B must lie between 0 and 1. The element positions are

d_1 = A=1/2 wavelength

d_2 = A+B = 0.5 + 0.707 = 1.207 wavelengths.

If an array of 4 elements is considered, the array factor is $\frac{1}{2}[\cos(d_1 v) + \cos(d_2 v)]$. For the above element positions, the first negative lobe is -0.4089 and this is the largest sidelobe in the pattern, as the design procedure predicts.

In the 5-element array, a +1 is introduced which does not occur in an array with an even number of elements. This decreases the magnitude of the large negative lobe and may cause a positive lobe to be the largest sidelobe in the pattern. For the 5-element array, the same element positions ($d_1 = \frac{1}{2}\lambda$, $d_2 = 1.207\lambda$) produce an array pattern with a negative sidelobe level of -0.1271 and a half-power beamwidth of 18°. However, the largest sidelobe is 0.267, a positive lobe caused by the +1 term. This causes a difficulty in design procedure, which must be considered, but it also lowers the sidelobe level of the 4-element array. The change from 0.4089 to 0.267 is quite significant.

Instead of writing the displacement condition, Equation 7a, in terms of the first negative sidelobe, one could write a restriction for the first positive sidelobe. This positive sidelobe occurs between v_3 and v_4 (see Fig. 2). The displacement equation for this lobe is

$$2\pi - Bv_3 = 2\pi B \quad . \tag{8a}$$

and for $v_3 = 2\pi/(A+B)$, one has

$$A = \frac{B^2}{1-B} \quad . \tag{8b}$$

The minimum value of A that places the positive peaks of the two cosine functions in the visible region is 1. Then by Equation 8b, B=0.618. The element positions are

$$d_1 = A = 1\,\lambda$$

$$d_2 = A+B = 1.618\,\lambda \ .$$

Fig. 3 shows a sketch of $\cos(d_1 v)$ and $\cos(d_2 v)$ for the above values. The complete array pattern is shown in Fig. 4. The maximum sidelobe level is -0.402 and the half-power beamwidth is about 12°. For A=1 then, the largest sidelobe occurs in the negative region instead of the assumed

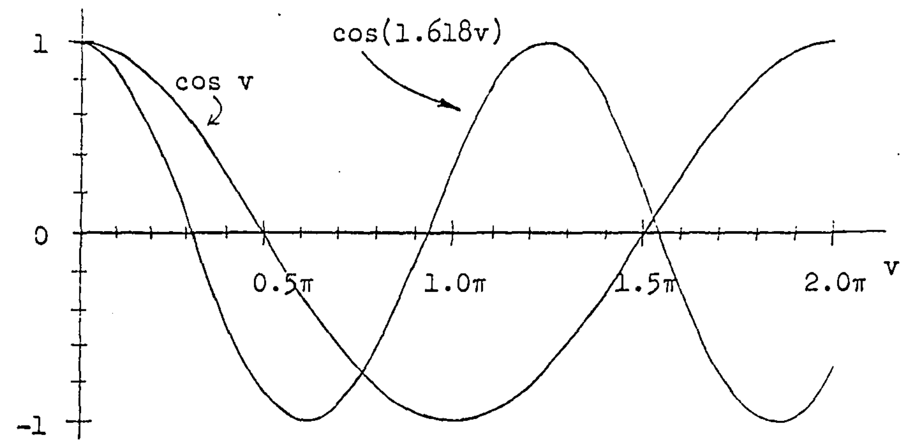

Fig. 3. Cosine functions for low sidelobes of a 5-element array

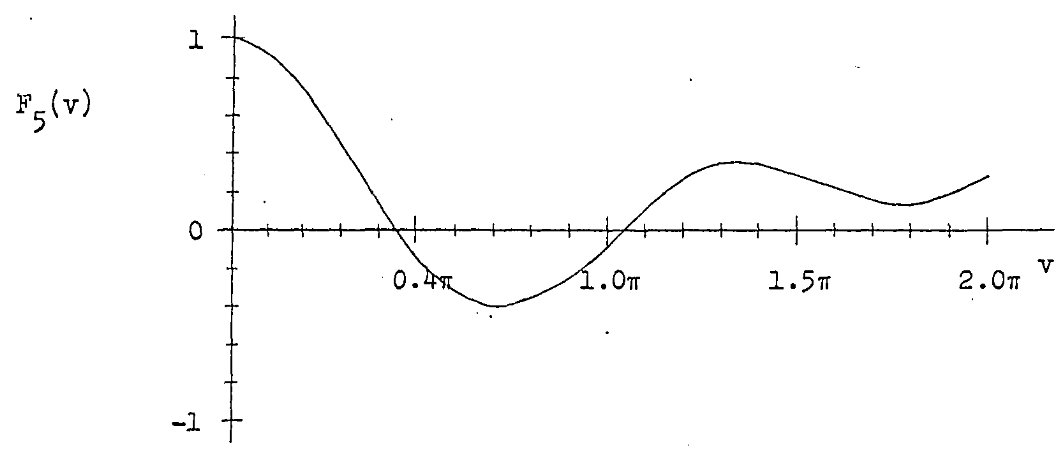

Fig. 4. Array pattern for low sidelobes of a 5-element array

positive region.

For low sidelobe values it is difficult to anticipate whether the first positive or first negative sidelobe will be the largest. The results shown here indicated that the best pattern was produced when the first negative sidelobe was assumed to be the greatest in the array with an odd number of elements. The lowest possible negative sidelobe level for a 5-element array is $\frac{1}{5}(1-2-2) = 0.6$ which would occur if the negative peaks of the cosine functions coincided. Thus, the patterns with the large sidelobe levels will definitely be caused by positive peaks of the cosine functions.

Of course, for the array with an even number of elements, this difficulty with the +1 term does not arise. The sidelobe largest in magnitude would always be the first negative lobe next to the main beam. It is worthwhile examining the array with an odd number of elements because the pattern is significantly improved with the addition of one element.

Comparing Figs. 3 and 4, one can see that a peak of the array pattern in Fig. 4 lies almost midway between two adjacent peaks of the functions sketched in Fig. 3. This provides a basis for designing an array with a prescribed sidelobe level.

ARRAY DESIGN

Five-Element Arrays

Five-element array design

When the value of A increases, the peaks of the cosine functions move closer to the origin. During this process the sidelobes increase, but the beamwidth decreases. Many applications require a narrow beamwidth as well as a low sidelobe level. These requirements are contradictory and a compromise must be made. In the formulation used here it is difficult to write a meaningful expression for the beamwidth in terms of the sidelobe level. The simplest procedure is to design an array with a specified sidelobe level and then determine the beamwidth by calculating the antenna pattern. This presents no difficulty with a high-speed computer.

The point where the maximum sidelobe occurs is represented as v_m. As previously stated, this can be approximated as the midpoint between the two positive peaks v_3 and v_4. Thus let

$$v_m \approx \frac{v_3 + v_4}{2} \quad . \tag{9}$$

If the two cosine functions had the same period, this expression for v_m would be exact. The peak value for the sum of the equal period curves would be greater than the peak value for the sum of the unequal period curves since the point of intersection is lower in the unequal period case. Equation 9 would give v_m for a "worst case" situation. In the previous example where $d_1 = 1$ and $d_2 = 1.618$, Equation 9 gives v_m as

$$v_m = \left(\frac{2\pi}{A+B} + \frac{2\pi}{A}\right)\frac{1}{2} = \pi\left(\frac{1}{1.618} + \frac{1}{1}\right)$$
$$= 1.618\ \pi$$

and the value of v_m from a computer calculation of the pattern is $1.36\,\pi$. The agreement is not exact, but as the sidelobes are allowed to increase the approximation becomes more accurate.

For a simple design procedure, there should be a convenient relationship between the specified sidelobe level and one of the element positions. This can be obtained if the two cosine functions are approximated as being equal at the highest sidelobe location. At $v=v_m$, $\cos(Av_m)$ is greater than $\cos(A+B)v_m$, so $\cos(Av_m) + \cos(A+B)v_m$ is approximated as $2\cos(Av_m)$ for a worst case situation. The value of the sidelobe at v_m for the previous example is estimated as

$$\frac{1}{5}(1 + 4 \cos Av_m) = \frac{1}{5}[1 + 4 \cos(1.618\pi)] = 0.49 \quad .$$

The computer-calculated pattern gives a maximum of 0.402. The actual value is seen to be a little less than the estimated value.

In the design problem, the maximum sidelobe value of $F_5(v)$ is specified and the spacings d_1 and d_2 are to be determined. The relationship between the given sidelobe level and an element position is

$$F_5(v_m) = \frac{1}{5}[1 + 4 \cos(Av_m)]$$

or

$$\cos(Av_m) = \frac{5\, F_5(v_m) - 1}{4} \quad . \tag{10}$$

From Fig. 2 it can be seen that Av_m lies in the fourth quadrant for the case where the first positive sidelobe is the highest sidelobe. For convenience let

$$Av_m = 2\pi - y\pi \qquad\qquad 0 < y < \frac{1}{2}$$

Substituting Equation 9 into the above expression gives

$$A\left(\frac{v_3 + v_4}{2}\right) = \pi(2-y)$$

$$A\pi\left(\frac{1}{A+B} + \frac{1}{A}\right) = \pi(2-y)$$

or $$B = \frac{Ay}{1-y} \quad . \tag{11}$$

Equations 8b and 11 completely specify the element positions when y is specified by the given sidelobe level. An example will illustrate the design procedure.

Design example for a 5-element array

It is given that a 5-element array is to have no sidelobe exceeding 0.6. Determine the element spacings.

By Equation 10, $\cos(Av_m) = 0.5$. Then $Av_m = 2\pi - \pi/3$ and $y = 1/3$. Equation 11 gives $B = A/2$ and Equation 8b becomes

$$A = \frac{A^2/4}{1-A/2} \quad .$$

The designed spacings are:

$$d_1 = A = 4/3$$

$$d_2 = A+B = 4/3 + 2/3 = 2 \quad .$$

The individual cosine functions are shown in Fig. 5 and a plot of $F_5(v)$ for this example appears in Fig. 6. The actual sidelobe level is 0.483 and the half-power beamwidth is about $9\frac{1}{2}°$. The sidelobe level is noticeably below the design level because of the worst case approximation made earlier.

The highest sidelobe in this example occurred at $v_m = 1.12\ \pi$. The estimated value of v_m is

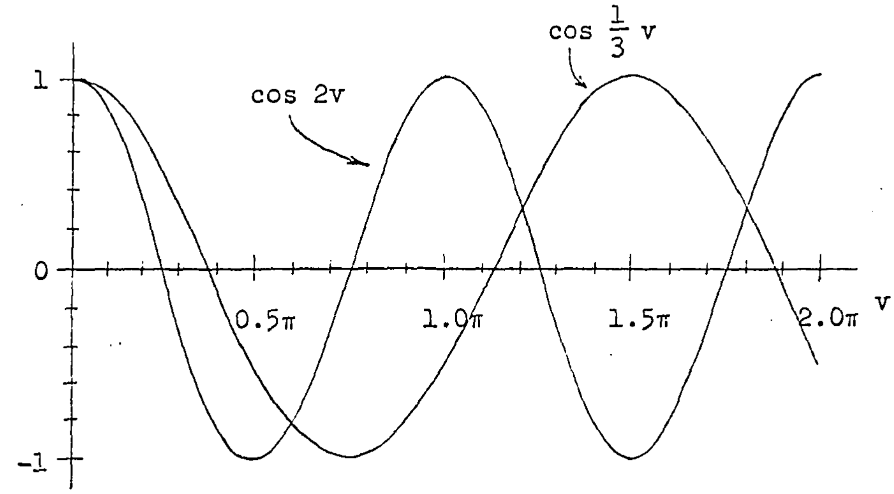

Fig. 5. Cosine functions for a designed sidelobe level of 0.6 for a 5-element array

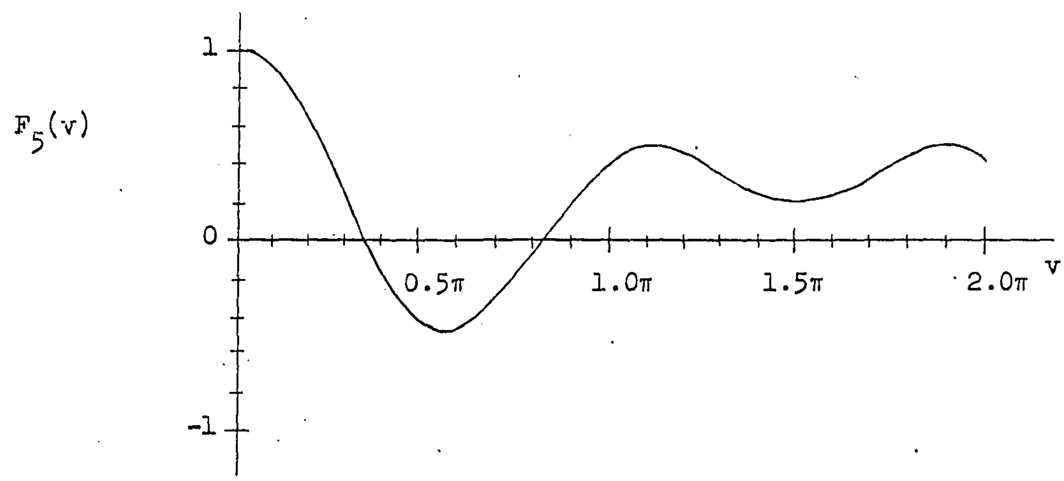

Fig. 6. Array pattern for a designed sidelobe level of 0.6 for a 5-element array

$$v_m = \frac{2\pi - y\pi}{A} = 1.25\ \pi \quad .$$

The agreement between the two values is quite acceptable.

A 5-element uniformly spaced array with a beamwidth of $9\frac{1}{2}°$ would have a grating lobe at $\theta = 63\frac{1}{2}°$. Placing the elements nonuniformly distributes the power that would be transmitted or received in the grating lobe region throughout the entire lobe region. This necessarily will increase the level of the sidelobes. However, in many situations this is more desirable than the grating lobes.

Seven-Element Arrays

The procedure used in the design of 5-element arrays can easily be applied to 7-element arrays. The array factor is expressed as

$$F_7(v) = \frac{1}{7}(1 + 2\cos d_1 v + 2\cos d_2 v + 2\cos d_3 v) \tag{12}$$

Let d_1=A, d_2=A+B, and d_3=A+B+C with A, B, and C greater than zero. A typical plot of the cosine functions is shown in Fig. 7. A displacement restriction is placed on the three functions so that no sidelobe is greater than the first large sidelobe. For the very low sidelobe case, the first negative peak is the source of the greatest lobe in the 7-element array. In addition to the restriction placed upon A and B as in the 5-element array, C is constrained as follows:

$$2\pi - v_3 C = 2\ C.$$

Replacing v_3 by $\pi/(A+B+C)$ gives

$$2\pi - \frac{C\pi}{A+B+C} = 2\pi C$$

or

$$A+B = \frac{C(2C-1)}{2(1-C)} \quad .$$

The two displacement relationships for the 7-element array then are

$$A = \frac{B(2B-1)}{2(1-B)} \tag{13a}$$

$$A+B = \frac{C(2C-1)}{2(1-C)} \tag{13b}$$

Minimum sidelobes

The minimum sidelobe level that is still regulated by the theory occurs near A=0.8, as determined by trial and error. For smaller values of A, two negative lobes of $\cos(d_3 v)$ occur before one negative peak of $\cos(d_1 v)$. Under these circumstances the maximum sidelobe does not occur where it is predicted. Lower sidelobes can be obtained for A less than 0.8, but the maximum sidelobe usually occurs near the end of the pattern. For A=0.8, Equations 13 give B=0.957 and C=0.827. The element positions are

$$d_1 = A = 0.8\ \lambda$$

$$d_2 = A+B = 1.557\ \lambda$$

$$d_3 = A+B+C = 2.384\ \lambda\quad .$$

A plot of the array factor is shown in Fig. 8. The first negative sidelobe is -0.2272 and occurs at v_m=0.520π. A higher sidelobe equal to -0.2496 occurs at v=2π .

Seven-element array design

There are two equations in three unknowns for the positions in the 7-element array. An additional equation is available that relates the specified sidelobe level to the element positions. This equation is

$$\frac{7\ F_7(v_m) - 1}{2} = \cos d_1 v_m + \cos d_2 v_m + \cos d_3 v_m \tag{14}$$

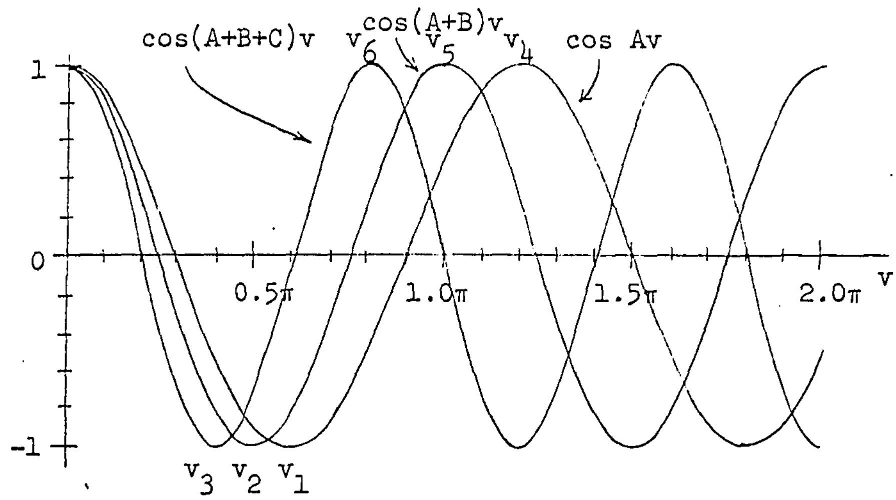

Fig. 7. Cosine functions for an arbitrary 7-element array

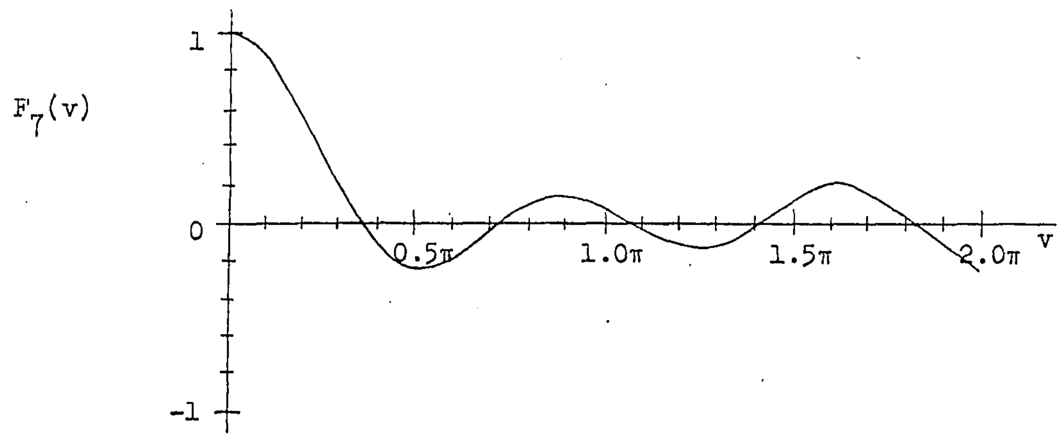

Fig. 8. 7-element pattern for low sidelobes

The expression is not very useful in this form. Good results can be obtained by simplifying the equation with some approximations. These are

$$v_m = v_2 \quad .$$

$$\cos d_1 v_m = \cos d_3 v_m \quad .$$

$$\cos d_2 v_m = -1 \quad .$$

With these approximations Equation 14 reduces to

$$\frac{7\ F_7(v_m) - 1}{2} = -1 + 2 \cos d_1 v_m$$

or

$$\frac{7\ F_7(v_m) - 1}{4} = \cos d_1 v_2 = \cos Av_2 \ . \qquad (15)$$

Av_2 will lie in the second quadrant and can be expressed as

$$Av_2 = \pi - y\pi \quad . \qquad 0 < y < \frac{1}{2}$$

From this, an expression relating A and B in terms of y can now be found.

$$A \frac{\pi}{A+B} = \pi(1-y)$$

$$B = \frac{Ay}{1-y} \quad . \qquad (16)$$

Equations 13 and 16 completely determine the element positions when $F_7(v_m)$ is specified.

<u>Design example for a 7-element array</u>

Determine the element spacings of a 7-element array for a maximum sidelobe level of 0.6.

For Equation 15, $\cos(Av_2) = -0.8$ and $y = 0.2$. By Equation 16, $B = 0.25A$. Using this value for B with Equations 13 one gets $A = 3.6$, $B = 0.9$, and $C = 0.915$. The element positions are

$$d_1 = A = 3.6\ \lambda$$

$$d_2 = A+B = 4.5\ \lambda$$

$$d_3 = A+B+C = 5.415\ \lambda\ .$$

The estimated value of v_m is

$$v_m = v_2 = \frac{\pi}{A+B} = 0.222\ \pi\ .$$

From the calculation of the antenna pattern, $v_m = 0.220\ \pi$, the sidelobe level is 0.605, and the half-power beamwidth is 3.5°. The predicted values of v_m and sidelobe value at v_m agree quite well with the pattern results. However, a positive maximum of 0.612 occurs at $v = 0.43\ \pi$ and a second positive maximum of 0.637 occurs at $v = 1.79\ \pi$. This is another case where the +1 term in the array factor has caused a positive lobe to dominate the first negative sidelobe. However, the results are quite satisfactory. A plot of $F_7(v)$ appears in Fig. 9. A uniformly spaced array with a beamwidth of 3.5° would have 2 grating lobes in the visible region, one at $\theta = 28°.9$ and a second at $\theta = 72°$.

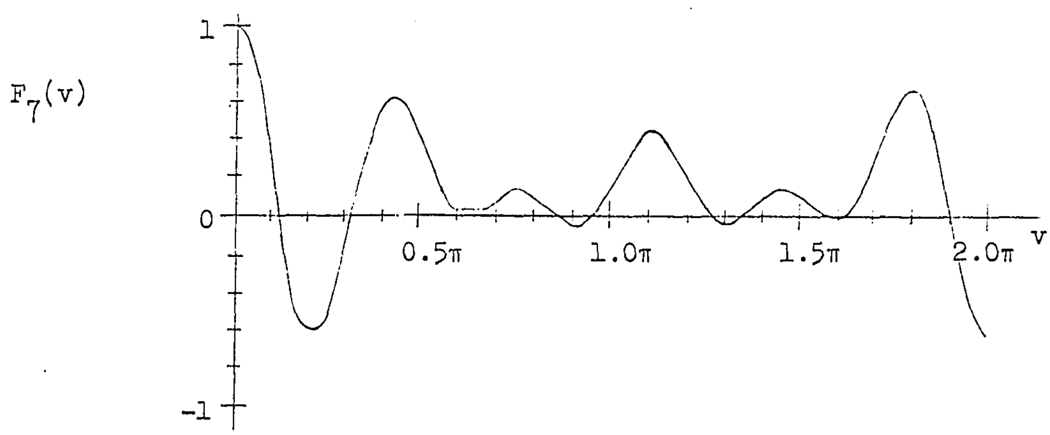

Fig. 9. Array pattern for a 7-element array with a designed sidelobe level of 0.6

Nine-Element Arrays

The extension of the displacement relationships to larger arrays is not difficult. For 2N+1 elements there are N-1 equations in N unknowns. An additional equation comes from the sidelobe specification. As the number of elements in the array increases, it becomes more and more difficult to use the sidelobe specification analytically. An example design problem will be worked for the 9-element array and there will then be a discussion of larger arrays.

The element positions for the 9-element array are denoted as d_1=A, d_2=A+B, d_3=A+B+C, and d_4=A+B+C+D where A, B, C, and D are greater than zero. For the first negative peak as the largest sidelobe, the displacement relations are

$$A = \frac{B(2B-1)}{2(1-B)} \tag{17a}$$

$$A+B = \frac{C(2C-1)}{2(1-C)} \tag{17b}$$

$$A+B+C = \frac{D(2D-1)}{2(1-D)} \quad . \tag{17c}$$

<u>Minimum sidelobes</u>

For the 9-element array, the smallest predictable lobes occur for A near 0.800, as determined by trial and error. Considering A=0.800, the remaining positions can be found from Equations 17a, 17b, and 17c respectively. The element positions are

$$d_1 = A = 0.800$$

$$d_2 = A+B = 1.557$$

$$d_3 = A+B+C = 2.384$$

$$d_4 = A+B+C+D = 3.250 \quad .$$

The array function is plotted in Fig. 10. The first negative sidelobe is -0.2039 and occurs at $v_m=0.40\pi$. The half-power beamwidth is $7^\circ\ 10'$. The largest sidelobe is 0.2249 which is due to the +1 term in the array factor.

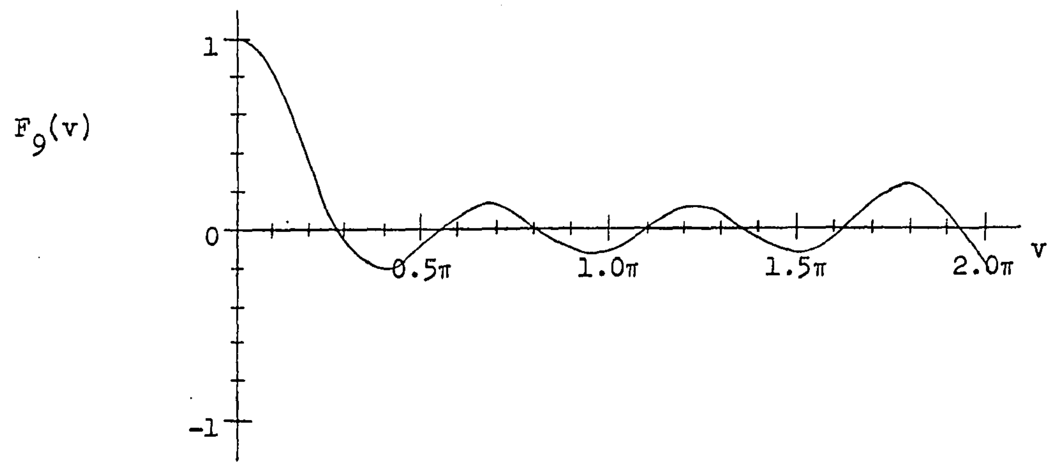

Fig. 10. 9-element pattern for low sidelobes

Design example for a 9-element array

If the maximum value of the sidelobes, $F_9(v_m)$ is specified, the requirement that must be satisfied is

$$\frac{9\ F_9(v_m) - 1}{2} = \cos d_1 v_m + \cos d_2 v_m + \cos d_3 v_m + \cos d_4 v_m . \quad (18)$$

It is desirable to find a simple relation between $F_9(v_m)$, v_m and one of the d's so that Equations 17 can be used to find the remaining distances. For this purpose, let

$$\cos d_1 v_m = \cos d_4 v_m$$

$$\cos d_2 v_m = \cos d_3 v_m \quad .$$

Then

$$\frac{9\,F_9(v_m) - 1}{4} = \cos d_1 v_m + \cos d_2 v_m \quad .$$

This is simpler but still is not convenient since two cosine functions are on the right hand side of the equation. To reduce this to one cosine function, set $\cos(d_2 v_m) = -1$ since it is nearly equal to this value. With these approximations, Equation 18 is reduced to

$$\frac{9\,F_9(v_m) - 1}{4} = \cos d_1 v_m - 1$$

or

$$\frac{9\,F_9(v_m) + 3}{4} = \cos A v_m \quad . \tag{19}$$

A 9-element array will now be designed so that no sidelobe exceeds a normalized value of 0.5.

The first step is to find Av_m from Equation 19. Since $F_9(v_m) = -0.5$, the value of $\cos(Av_m)$ is -0.375. $Av_m = 0.6241\,\pi$ which is in the second quadrant. To relate v_m to the parameters A, B, C, and D, let v_m lie midway between $\cos(d_2 v_m)$ and $\cos(d_3 v_m)$. That is, let

$$v_m = \frac{1}{2}\left(\frac{\pi}{A+B} + \frac{\pi}{A+B+C}\right) \quad .$$

Equate this to $v_m = 0.6241\,\pi/A$ as was found from the sidelobe restriction.

$$\frac{0.6241}{A} = \frac{1}{2}\left(\frac{1}{A+B} + \frac{1}{A+B+C}\right) \quad .$$

This simplifies to

$$1.2482\,(A+B)(A+B+C) = A(A+B+C) + A(A+B) \quad . \tag{20}$$

From Equation 17b one can write

$$A+B+C = \frac{C(2C-1)}{2(1-C)} + C = \frac{C}{2(1-C)} \quad . \tag{21}$$

Equations 17b and 20 are substituted into Equation 21 and then reduced to

$$C = \frac{A + 0.312}{A + 0.624} \quad . \tag{22}$$

Next, substitute this into Equation 17b. The relationship between A and B is

$$B = \frac{0.6026\ A - 0.124}{A + 0.624}\ A \ .$$

When this is placed into Equation 17a, the resultant equation in A is

$$A^3 - 1.2413\ A^2 - 1.4870\ A - 0.3632 = 0 \ .$$

The only real root of this equation is A = 2.052. The remaining spacing parameters are easily found from Equations 17. The element positions are:

$$d_1 = A = 2.052$$

$$d_2 = A+B = 2.9052$$

$$d_3 = A+B+C = 3.7886$$

$$d_4 = A+B+C+D = 4.6923 \quad .$$

The estimated value of v_m is

$$v_m = \frac{\pi}{2}\left(\frac{1}{A+B} + \frac{1}{A+B+C}\right) = 0.304\ \pi \quad .$$

The array factor is plotted in Fig. 11. The actual value of v_m is $0.28\ \pi$ and the sidelobe level at this point is 0.466. These values compare very favorably with the estimated values. While $\cos(d_2 v_m)$ is -0.833 rather than the approximation of -1 made earlier to simplify the transcendental equation, the close comparison between the designed value and the actual

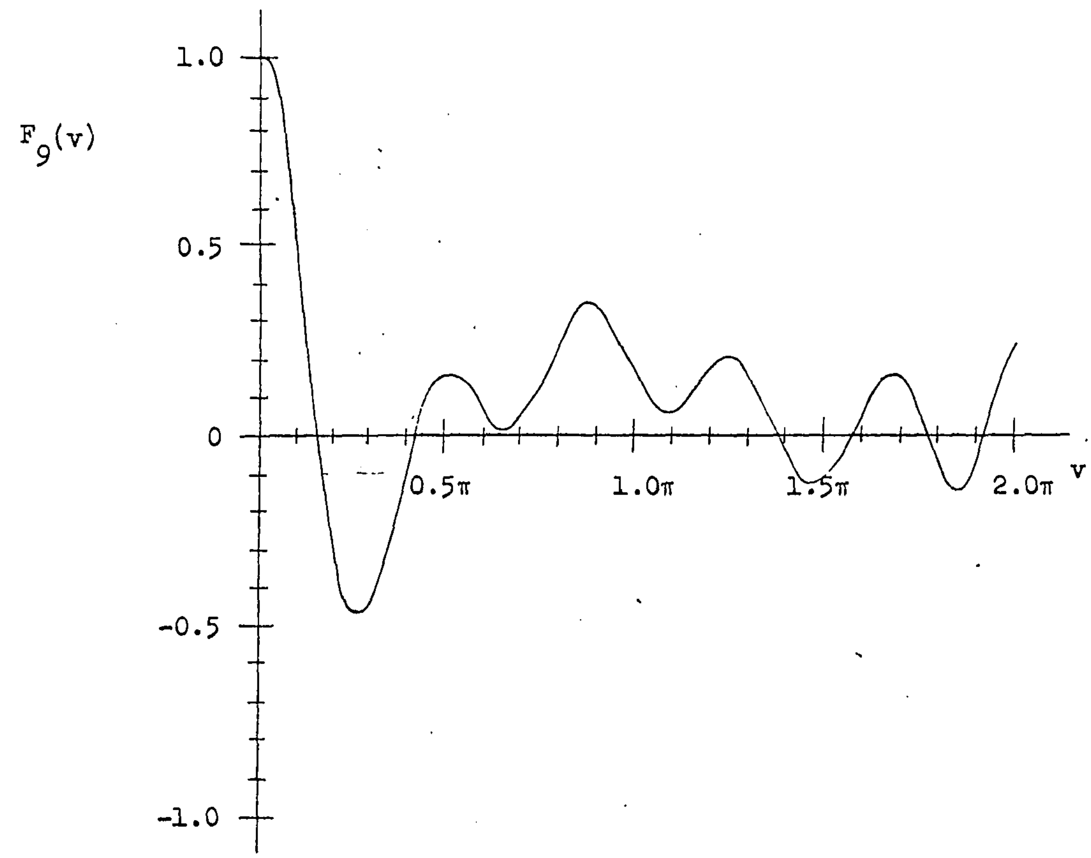

Fig. 11. Array pattern for a 9-element array with a designed sidelobe level of 0.5

value justifies the procedure. The half-power beamwidth is about $4\frac{1}{2}°$. A uniformly spaced array with this beamwidth would have a grating lobe at $50°21'$.

It is clear from these results that one of the advantages of spacing the elements nonuniformly is to eliminate the grating lobes that would appear in a uniformly spaced array. The spacing between adjacent elements has been less than one wavelength (except for d_1) which makes it difficult to compare the arrays presented here with the 9-element arrays calculated by Lo and Lee [42]. Their arrays had an average spacing of 2.375 wavelengths over an aperture of 19.5 wavelengths. The sidelobes are higher than those produced here because they spread two grating lobes over the entire sidelobe region and only one grating lobe was dispersed here.

The procedure demonstrated in the 5-, 7-, and 9-element array designs should give a basic understanding of the cosine displacement method of array design. A summary of the design method follows.

Summary of Design Procedure

For a specified sidelobe level, the first step is to relate this numerical value to an element position. Various procedures were shown for doing this in the 5-, 7-, and 9-element array examples. All procedures had the objective of reducing the number of cosine functions in the transcendental equation to only one in the region of the maximum sidelobe. The argument of the resulting cosine function contained only the first element position.

The displacement equations, used with the equation containing the sidelobe specification, are sufficient to completely specify the desired element positions. The first position, d_1, is used to solve for d_2, d_2 used to find d_3, and d_n found by using d_{n-1}. Each expression for an element position is a quadratic equation and can be solved very simply. One root of the equation produces a negative value for $d_n - d_{n-1}$ which has been prohibited by previous constraints. Thus, there is no ambiguity in the element positions.

The beamwidth is not calculated during the design procedure, but is found from the synthesized array pattern. If this beamwidth is not narrow enough to meet the design requirement, the sidelobe level must be allowed to increase, and then new element positions calculated.

POSSIBLE EXTENSIONS OF THE COSINE DISPLACEMENT THEORY

Spacings Larger than One Wavelength

If the individual elements in the array are nonisotropic, the element positions can be increased beyond one wavelength. Consider an array of elements, each with a beamwidth of 5° ($\theta=2\frac{1}{2}^\circ$), and low sidelobe levels beyond the main beam. A grating lobe could exist in the isotropic element array at $\theta = 5^\circ$ and not cause any change in the sidelobe level of the nonisotropic element array. If the distance between adjacent cosine peaks at $\theta = 10^\circ$ ($\pi/18$ radians) is the same as the distance between adjacent peaks at the first negative sidelobe, the displacement equation is

$$2\,\pi - B\frac{\pi}{A+B} = B\frac{\pi}{18}$$

which reduces to

$$A = \frac{B(B-18)}{36-B} \quad .$$

B can now range from 18 wavelengths to 36 wavelengths, which is very useful from a practical standpoint. An element with a beamwidth of 5° is probably larger than one wavelength and a large spacing is required for physical realizability of the array. Also, mutual coupling is reduced. However, an array designed under these conditions is almost certain to have a grating lobe in the neighborhood of 10°, and one must be assured that this is acceptable when undertaking this design procedure.

Arrays Larger than Nine Elements

From the difficulty encountered in reducing the transcendental equation 18 to a workable form, one can imagine that for larger arrays the task is even worse. Since there is only one equation lacking in the set of displacement equations, it becomes questionable whether the work involved in reducing the transcendental equation to a useful form is worth the effort. One of the objectives of the design procedure presented here is to keep the method as simple as possible.

For larger arrays it may be much easier to use a graphical technique than an analytic technique for obtaining the additional information required to solve the displacement equations for the element positions. Array patterns were calculated for 4 17-element arrays to see if a plot of element-position d_1 (or A) vs. sidelobe level would provide the information needed in a simple manner. This plot is shown in Fig. 12, and a plot of beamwidth vs. sidelobe level is shown in Fig. 13. The solid curve shows the highest negative lobe, and the dotted curve the highest sidelobe level in the entire array pattern.

The relationship between A and the sidelobe level is very nearly linear, and it appears that this could be used very nicely for array design. To test this hypothesis, a sidelobe level of 0.35 was chosen for a design. Using the linear relationship, A was found to be 2.025. The remaining positions are easily found from the displacement relations.

The calculated sidelobe level for this array is 0.358 and the beamwidth is $2°44'$. These values are very close to those obtained from the graphs. This appears to be a very simple and effective method for array designs. No further graphical work has been completed and it is not known whether the linear relationship between position d_1 and the sidelobe level extends to larger arrays.

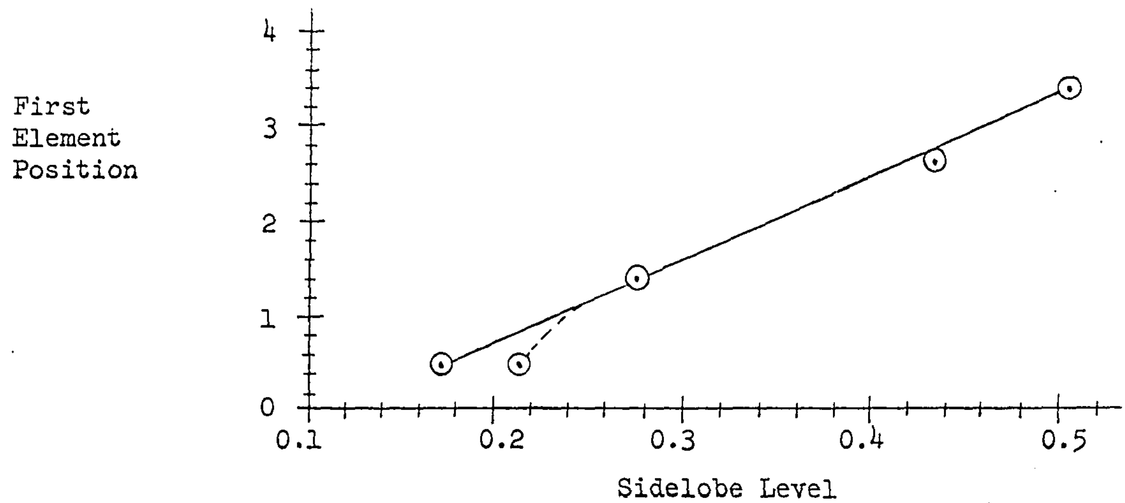

Fig. 12. Element position and sidelobe level relationship for a 17-element array

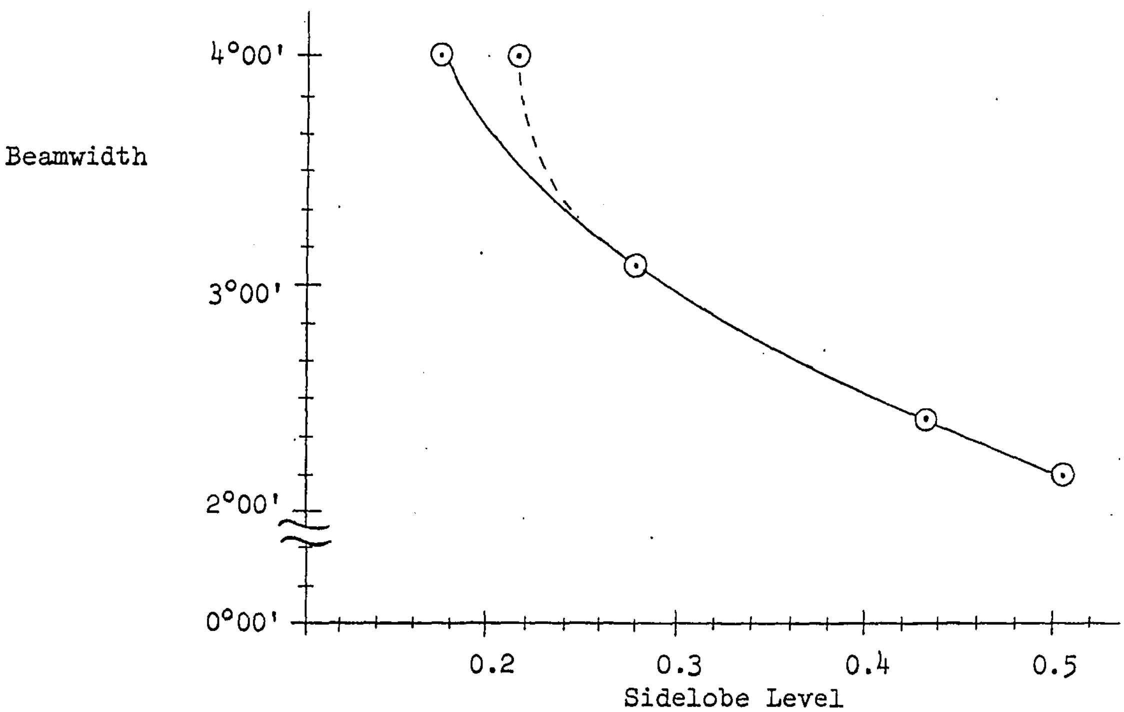

Fig. 13. Half-power beamwidth and sidelobe level relationship for a 17-element array

SUMMARY AND CONCLUSIONS

Many techniques have been developed for analyzing and synthesizing nonuniformly spaced arrays. The sidelobe levels in many of the designs are quite low near the center of the array pattern, but increase greatly near the outer regions. Procedures that have been developed to give a rather constant sidelobe level have a higher overall level than the region between grating lobes of uniformly spaced arrays. This higher level is expected since energy in the grating lobes has been dispersed over the entire array pattern. The techniques are frequently difficult to apply and the results often do not justify the time spent in the design.

The technique presented here is very easy to use for rapid effective designs of small or medium sized nonuniformly spaced arrays. No sophisticated computation facilities are required to obtain the element positions and the results appear to be excellent. If more-optimum arrays are desired, this method provides good insight on pattern behavior as a function of element spacing and can serve as a starting point for computer solutions. The synthesized patterns were shown to be better than those of uniformly spaced arrays that contained grating lobes.

The design method is easily extended to arrays with nonisotropic radiators. The element spacing increases for these designs which allows greater freedom than small spacings for physical construction of the array.

Preliminary calculations have shown that it may be possible to apply graphical design techniques to arrays in the intermediate size category of 20 or 30 elements. A graph could be made easily and quickly that contained information of several array designs for a given number of elements. The specifications taken from the graph can be used to conveniently obtain the final design. Several trials are often necessary in array design, and with a simple design method very little additional effort is expended in making a graph.

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